HERMITIAN METRICS.

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Introduction. The smoke of battle that used to surround the various Non-Euclidean geometries has passed away and we are at present able to appreciate the underlying significance of much that was formerly mysterious and profited, to a certain extent, by the charm that attaches to mystery. To the abstract logician the Non-Euclidean problem comes under the general head of the independent axiom problem. To the working mathematician, assuming for the sake of argument that such a person exists, the matter presents itself in about the following terms:

"Here is a collection of objects which we call points. Each is possessed of a set of coördinates, we do not care whether they were honestly acquired or not. Here is a function which we call the distance of two points. What can be said about the system?"

In the classical Non-Euclidean systems the function is obtained by polarizing a given quadratic form, and these geometries will doubtless always remain the most important; but endless other methods of procedure are conceivable, and all are not entirely barren of interesting results. Perhaps the most successful venture of this sort consists in using, as the basis of measurement, a so-called "Hermitian form." This is an expression of the type

$$\Sigma a_{ij} x_i \bar{x}_j \qquad a_{ji} = \bar{a}_{ij} \qquad (i, j = 0, 1, \cdots n),$$

where any letter with a dash over the top is supposed to be the conjugate imaginary of the same letter without the dash. It is usual also to consider only the case where the discriminant is not zero, and to define as the distance of two points (x) (y) the number d defined by the equation

$$\cos\frac{d}{k} = \frac{\sqrt{\Sigma a_{ij} x_i \bar{y}_j}\,\sqrt{\Sigma a_{ij} y_i \bar{x}_j}}{\sqrt{\Sigma a_{ij} x_i \bar{x}_j}\,\sqrt{\Sigma a_{ij} y_i \bar{y}_j}}.$$

It is not difficult to show that by a properly chosen collineation our Hermitian form can be reduced to the canonical type

$$\Sigma \pm x_i \bar{x} .$$

Of all possible geometries based upon such forms, the richest and most symmetrical is the so called "elliptic" type where all of the terms above

are positive. If we use the common abbreviation

$$(ab) = \Sigma a_i b_i$$

the fundamental distance formula is

$$\cos d = \frac{\sqrt{(x\bar{y})}\sqrt{(\bar{x}y)}}{\sqrt{(x\bar{x})}\sqrt{(y\bar{y})}}.$$

The next most interesting type is a limiting case of this. Let us replace d by d/k, and x_0 by kx_0. Then

$$k \sin\frac{d}{k} = \frac{\sqrt{\sum_{i=0}^{i=n}(x_i y_0 - x_0 y_i)(\bar{x}_i\bar{y}_0 - \bar{x}_0\bar{y}_i) + \frac{1}{2k^2}\sum_{i,j=1}^{i,j=n}(x_i y_j - x_j y_i)(\bar{x}_i\bar{y}_j - \bar{x}_j\bar{y}_i)}}{\sqrt{x_0\bar{x}_0 + \frac{1}{k^2}\left(\sum_{i=1}^{i=n} x_i\bar{x}_i\right)}\sqrt{y_0\bar{y}_0 + \frac{1}{k^2}\left(\sum_{i=1}^{i=n} y_i\bar{y}_i\right)}}$$

and the limit of this as k becomes infinite is

$$d = \frac{\sqrt{\sum_{i=1}^{i=n}(x_i y_0 - x_0 y_i)(\bar{x}_i\bar{y}_0 - \bar{x}_0 y_i)}}{\sqrt{x_0\bar{x}_0}\sqrt{y_0\bar{y}_0}},$$

which looks more familiar in the non-homogeneous form*

$$d = \sqrt{(x' - x)(\bar{x}' - \bar{x}) + (y' - y)(\bar{y}' - \bar{y}) + (z' - z)(\bar{z}' - \bar{z}) + \cdots}.$$

1. **The metric of a single line.** Let us fix a point on a line by two homgeneous coördinates x_0, x_1 not both simultaneously zero.

The distance of two points shall be given by the equation

$$\cos d = \frac{\sqrt{x_0\bar{y}_0 + x_1\bar{y}_1}\sqrt{\bar{x}_0 y_0 + \bar{x}_1 y_1}}{\sqrt{x_0\bar{x}_0 + x_1\bar{x}_1}\sqrt{y_0\bar{y}_0 + y_1\bar{y}_1}}. \tag{1e}$$

To find a geometrical meaning for this let us represent the totality of points of our line by the real points of a Riemann sphere†

$$X = \frac{x_0\bar{x}_1 + x_1\bar{x}_0}{x_0\bar{x}_1 + x_1\bar{x}_1}, \qquad Y = i\frac{x_0\bar{x}_1 - x_1\bar{x}_0}{x_0\bar{x}_0 + x_1\bar{x}_1}, \qquad Z = \frac{x_0\bar{x}_0 - x_1\bar{x}_1}{x_0\bar{x}_0 + x_1\bar{x}_1},$$

$$X^2 + Y^2 + Z^2 = 1.$$

* The present author is familiar with only two articles on the Hermitian metric. The first of these was a slight contribution by Fubini "Sulle metriche definite da una forma Hermitiana," Atti della R. Istituto Veneto, vol. LXIII, 1903–04. Far different is the fundamental article by Study "Kurzeste Wege im komplexen Gebiete," Math. Annalen, vol. 60, 1905. The parabolic case is here treated in a stepmotherly fashion, but the elliptic one receives long attention. Yet Study, curiously enough, does not touch upon the most immediate and elementary questions of the Hermitian metric, so that there is very little overlapping between his work and the present paper.

† We shall habitually use large letters to signify real values, and small ones for complex ones.

If the complex point y_0, y_1 corresponds to the real point (X', Y', Z') and Θ is the angle subtended at the center of the sphere by the points $(X, Y, Z)(X', Y', Z')$

$$\cos\frac{\Theta}{2} = \sqrt{\frac{XX' + YY' + ZZ' + 1}{2}}$$

$$= \frac{\sqrt{x_0\bar{y}_0 + x_1\bar{y}_1}\sqrt{\bar{x}_0 y_0 + \bar{x}_1 y_1}}{\sqrt{x_0\bar{x}_0 + x_1\bar{x}_1}\sqrt{y_0\bar{y}_0 + y_1\bar{y}_1}}$$

$$= \cos d.$$

THEOREM 1*e*. *The elliptic Hermitian distance of two points is one half the arc of the great circle connecting the corresponding points of the Riemann sphere.**

Passing over to the parabolic case we take the distance formula

$$d = \frac{\sqrt{(x_0 y_1 - x_1 y_0)}\sqrt{(\bar{x}_0\bar{y}_1 - \bar{x}_1\bar{y}_0)}}{\sqrt{x_0\bar{x}_0}\sqrt{y_0\bar{y}_0}} \tag{1p}$$

$$= \sqrt{(x' - x)(\bar{x}' - \bar{x})}.$$

If we write

$$x = X + iY, \qquad x' = X' + iY',$$

$$d = \sqrt{(X' - X)^2 + (Y' - Y)^2}.$$

THEOREM 1*p*. *The parabolic Hermitian distance of two points is the absolute value of their Euclidean distance, and also the distance of the two points which represent them in the Gauss plane.*

Let A, B, C be three points. When shall we have the equation

$$AC + CB = AB?$$

We see at once that, in the elliptic case, the real points which represent the given complex ones must lie on a great circle, and in the parabolic case the representing points must be collinear. Also AB must be the largest of the three distances.

Definition. A system of collinear points of such a nature that the cross ratio of any four is real, shall be said to belong to a *chain*.†

A chain connecting the points (X') and (X'') can always be represented in the form

$$x_i = \Xi_1 x_i' + \Xi_2 \rho x_i'', \qquad \bar{x}_i = \Xi_1 \bar{x}_i' + \Xi_2 \bar{\rho} \bar{x}_i'', \qquad i = 0, 1, \cdots n. \tag{2}$$

* Study, loc. cit., p. 333, writes $d/2$ where we write d, so as to identify distance on the complex line with arc on the Riemann sphere.

† The literature of chains is large. The original idea is due to Von Staudt. See his "Beiträge zur Geometrie der Lage," Part 2, Nuremberg, 1853 ±.

If, in the elliptic case, we have the additional relation

$$\bar{\rho}(x'\bar{x}'') = \rho(\bar{x}'x''),$$

the chain is said to be a *normal* one. The equations for a normal chain are simplified if we imagine the complex multiplier ρ swallowed into the homogeneous coördinates (x''). We have, then

$$\begin{gathered} x_i = \Xi_1 x_i' + \Xi_2 x_i'', \qquad \bar{x}_i' = \Xi_1 \bar{x}_i' + \Xi_2 \bar{x}_i'', \\ (x'\bar{x}'') = (\bar{x}'x''). \end{gathered} \tag{3}$$

In the one-dimensional elliptic case if we now put

$$\Xi_1' = (\bar{x}_0'x_0'' + \bar{x}_1'x_1'')\Xi_1 + (x_0''\bar{x}_0'' + x_1''\bar{x}_1'')\Xi_2,$$

$$\Xi_2' = -(x_0'\bar{x}_0' + x_1'\bar{x}_1')\Xi_1 - (x_0'\bar{x}_0'' + x_1'\bar{x}_1'')\Xi_2,$$

we carry our normal chain into itself while we replace x_0, x_1 by $\bar{x}_1$, $-\bar{x}_0$ and X, Y, Z by $-X$, $-Y$, $-Z$, so that each great circle is invariant under the transformation. Hence our normal chain corresponds to a great circle. For a normal chain in the parabolic case we write

$$x_0'\bar{x}_0'' = \bar{x}_0'x_0''.$$

We can find a point on the chain for which the first coördinate is zero, i.e., the chain contains the infinite point on the line, and is represented in the Gauss plane by a straight line.

THEOREM 2. *The necessary and sufficient condition that three collinear points should be connected by a relation*

$$AC + CB = AB$$

is that they should belong to one normal chain, and that the last of the three distances determined by them should be the greatest.

2. **Plane Trigonometry.** We next suppose that we are dealing with points in one plane. For the distance of two points we have, in the elliptic case

$$\cos d = \frac{\sqrt{(x\bar{y})}\sqrt{(y\bar{x})}}{\sqrt{(x\bar{x})}\sqrt{(y\bar{y})}}, \tag{4e}$$

whereas in the parabolic case we have

$$d = \sqrt{(x' - x)(\bar{x}' - \bar{x}) + (y' - y)(\bar{y}' - \bar{y})}. \tag{4p}$$

With regard to the first of these formulæ we notice that

$$1 - \cos^2 d = \frac{\frac{1}{2}\Sigma(x_iy_j - x_jy_i)(\bar{x}_i\bar{y}_j - \bar{x}_j\bar{y}_i)}{(x\bar{x})(y\bar{y})} > 0.$$

If the equation of a line be written in the compact form

$$(ux) = 0$$

we shall define the elliptic angle of two lines (u) and (v) by the equation

$$\cos\theta = \frac{\sqrt{(u\bar{v})}\sqrt{(\bar{u}v)}}{\sqrt{(u\bar{u})}\sqrt{(v\bar{v})}}. \tag{5e}$$

In the parabolic case we shall adopt the simpler form

$$\cos\theta = \frac{\sqrt{u_1\bar{v}_1 + u_2\bar{v}_2}\sqrt{\bar{u}_1v_1 + \bar{u}_2v_2}}{\sqrt{u_1\bar{u}_1 + u_2\bar{u}_2}\sqrt{v_1\bar{v}_1 + v_2\bar{v}_2}}. \tag{5p}$$

We shall define the distance from a point to a line, as the distance to the foot of the perpendicular on the line. We thus get, for the point (x) and the line (u) the twin formulas

$$\sin d = \frac{\sqrt{(ux)}\sqrt{(\bar{u}\bar{x})}}{\sqrt{(u\bar{u})}\sqrt{(x\bar{x})}}, \tag{6e}$$

$$d = \frac{\sqrt{(ux)}\sqrt{(\bar{u}\bar{x})}}{\sqrt{u_1\bar{u}_1 + u_2\bar{u}_2}\sqrt{x_0\bar{x}_0}}. \tag{6p}$$

For real elements, the elliptic form of Hermitian measurement is identical with that for a projective plane in elliptic space of total curvature unity, while the parabolic Hermitian measurement is identical with the usual Euclidean form.

A collineation which leaves distances invariant shall be said to be "congruent." An elliptic congruent collineation may be written*

$$\rho x_i' = a_ix_0 + \frac{[(\bar{a}b)a_i - (a\bar{a})b_i]}{\sqrt{(a\bar{a})(b\bar{b}) - (a\bar{b})(\bar{a}b)}}x_1 + \frac{\sqrt{(a\bar{a})}(a_jb_k - a_kb_j)}{\sqrt{(a\bar{a})(b\bar{b}) - (a\bar{b})(\bar{a}b)}}x_2 \tag{7e}$$

The parabolic collineation takes the somewhat simpler form

$$\begin{aligned} \rho x_0' &= a_0x_0 \\ \rho x_1' &= a_1x_0 + \cos Ae^{i\Theta_1}x_1 + \sin Ae^{i(\Theta_1+\Phi)}x_2, \\ \rho x_2' &= a_2x_0 \pm \sin Ae^{i\Theta_2}x_1 \mp \cos Ae^{i(\Theta_2+\Phi)}x_2. \end{aligned} \tag{7h}$$

We see that the point $(1, 0, 0)$ is carried into the arbitrary point (a). Hence, the transformation is transitive. On the other hand, if $(1, 0, 0)$

* The first writer to give implicitly the form for a congruent collineation of the elliptic type was Loewy. See his "Über bilineare Formen mit konjugirt imaginären Variabeln," Nova Acta Leopoldina, vol. 71, 1898. The explicit form, was first given by the author in his article "The Geometry of Hermitian Forms," Transactions American Mathematical Society, vol. 21, 1920.

is fixed, we have, in both cases, the same type of rotation, which will carry any line through the fixed point into any other such line. Lastly, if a point and a line through it be fixed (which fixes the perpendicular through the point), there is still possible a group of congruent collineations depending on two real parameters namely

$$\rho x_i' = e^{\Theta_i} x_i. \tag{8}$$

Definition. A system of concurrent and coplanar lines of such a sort that the cross ratio of any four is real is said to belong to a "line chain."

The general analytic expression for a chain will be:

$$u_i = H_1 u_i' + H_2 \rho u_i''. \tag{9}$$

This chain, in the elliptic case, will be defined as *normal* if

$$\bar{\rho}(u'\bar{u}'') = \rho(\bar{u}'u''). \tag{10e}$$

Since distance and angle formulas are the same in the elliptic case, this shows that, if a, b, c be three lines of the chain, there will always be a relation of the form:

$$\angle\, ac + \angle\, cb = \angle\, ab.$$

A similar result will hold in the parabolic case if the terms with subscript zero be suppressed. A line chain will always cut a transversal in a chain of points; when will a normal line chain cut a normal point chain? Let the transversal be the line $x_2 = 0$, while the vertex of the line pencil is $(1, 0, c)$. A typical line of the chain may be written

$$H_1(acx_0 - cx_1 - ax_2) + H_2\rho[bcx_0 - cx_1 - bx_2] = 0.$$

Since the line chain is by hypothesis, normal, we have in the elliptic case

$$\bar{\rho}[c\bar{c}(a\bar{b} + 1) + a\bar{b}] = \rho[c\bar{c}(\bar{a}b + 1) + \bar{a}b].$$

On the other hand, the point chain is expressed

$$x_0 = H_1 c + H_2 \rho c, \qquad x_1 = H_1 ac + H_2 \rho bc;$$

and this will be normal if

$$\bar{\rho} c\bar{c}[a\bar{b} + 1] = \rho c\bar{c}[\bar{a}b + 1].$$

The two conditions are compatible when, and only when

$$\rho = \bar{\rho}, \qquad a\bar{b} = \bar{a}b.$$

Exactly similar reasoning is applicable in the parabolic case, and we have

THEOREM 3. *The necessary and sufficient condition that a normal line chain should cut a transversal in a normal chain, is that the perpendicular*

from the vertex of the line chain upon the transversal should be included in the line chain. The necessary and sufficient condition that a normal point chain should determine a normal chain about a given point, is that the foot of the perpendicular from the given point upon the line of the chain should belong to the line chain.

The two equations just written express the necessary and sufficient condition that there should exist a congruent transformation of the type (8) which carries the three sets of coördinate values $(1, 0, c)$, $(c, ac, 0)$, $(\rho c, \rho bc, 0)$, into three real sets:

THEOREM 4. *A normal line chain will intersect a transversal in a normal point chain, and a normal point chain will determine a normal line chain about a given point, when, and only when, there exists a congruent collineation which transforms the two simultaneously into real point and line chains.*

We shall presently see that the perpendicular from a point upon a line acts as a universal solvent in many trigonometric problems. Let us first look for the formulas for the right triangle. By a proper change of axes we may suppose that if we have a triangle right-angled at C we may take for the coördinates of the vertices $(1, 0, 0)$, $(1, a, 0)$, $(1, 0, c)$; while the sides have the coördinates $(-ac, c, a)$, $(0, 0, 1)$, $(0, 1, 0)$. In the elliptic case

$$\cos AB = \frac{1}{\sqrt{1 + a\bar{a}}\sqrt{1 + c\bar{c}}} = \cos BC \cos CA,$$

$$\sin BC = \frac{\sqrt{c\bar{c}}}{\sqrt{1 + c\bar{c}}} = \frac{\sqrt{a\bar{a}c\bar{c} + a\bar{a} + c\bar{c}}}{\sqrt{1 + a\bar{a}}\sqrt{1 + c\bar{c}}} \cdot \frac{\sqrt{c\bar{c} + a\bar{a}c\bar{c}}}{\sqrt{a\bar{a}c\bar{c} + a\bar{a} + c\bar{c}}}$$

$$= \sin AB \sin A.$$

Exactly similar formulas hold in the parabolic case:

THEOREM 5. *The trigonometric formulas for a right triangle are the same in the Hermitian metrics as in the corresponding Non-Euclidean or Euclidean metrics.*

We see that the hypotenuse of a right triangle is always greater than one leg, and a side of a triangle is greater than its projection on another side:

THEOREM 6. *If A, B, C, be any three points the distance AB is never greater than the sum of the distances AC and CB and is only equal to that sum when (a) the points are collinear, (b) they belong to a normal chain, and (c) AB is the greatest of the three distances.**

The normal chain is thus a geodesic in our plane, i.e., the shortest path between two given points.

* For an algebraic proof see Study, loc. cit., p. 330ff.

If an arbitrary triangle be given, we may take as the coördinates of its vertices the values $(1, 0, c)$, $(1, a, 0)$, $(1, b, 0)$. These points may be carried simultaneously by a congruent collineation into real points when, and only when

$$a\bar{b} = \bar{a}b.$$

When this relation holds, it is evident that the usual elliptic or Euclidean trigonometric relations must hold between the sides and angles of the triangle. Conversely, if such relations hold, the foot of an altitude must lie on a normal chain with the two vertices collinear therewith, and we can assume that the vertices have these coördinates and that this equation is true. We may, however, put the matter still more concretely. Two sides of our triangle have the equations

$$-acx_0 + cx_1 + ax_2 = 0, \qquad -bcx_0 + cx_1 + bx_2 = 0.$$

The perpendiculars on these from the opposite vertices have, in the elliptic case, the equations

$$a\bar{b}x_0 - \bar{b}x_1 + \bar{c}(1 + a\bar{b})x_2 = 0, \qquad \bar{a}bx_0 - \bar{a}x_1 + \bar{c}(1 + \bar{a}b)x_2 = 0.$$

These will be concurrent upon the third altitude when, and only when

$$a\bar{b} = \bar{a}b.$$

THEOREM 7. *The necessary and sufficient condition that the Hermitian trigonometry of a triangle be that of the corresponding Non-Euclidean or Euclidean system of metrics, is that the altitudes should be concurrent. In this case, and in this case only, the foot of one altitude, and, hence of each altitude, is on the normal chain determined by the vertices collinear with it. In this case, and in this case only, it is possible to carry the triangle by a congruent collineation into a real triangle.*

3. **Hyperconics.** Let us find the locus of a point at a given distance from a given point (y). In the elliptic case we have, clearly

$$(x\bar{y})(\bar{x}y) - \cos^2 d(x\bar{x})(y\bar{y}) = 0,$$

and in the parabolic one

$$(x_1y_0 - x_0y_1)(\bar{x}_1\bar{y}_0 - \bar{x}_0\bar{y}_1) + (x_2y_0 - x_0y_2)(\bar{x}_2\bar{y}_0 - \bar{x}_0\bar{y}_2) - d^2x_0\bar{x}_0y_0\bar{y}_0 = 0,$$

which is written non-homogeneously

$$(x' - x)(\bar{x}' - \bar{x}) + (y' - y)(\bar{y}' - \bar{y}) = d^2.$$

In the former case, if $d = 0$ we have

$$\Pi(x_iy_j - x_jy_i)(\bar{x}_i\bar{y}_j - \bar{x}_j\bar{y}_i) = 0$$

so that the point (y) alone lies on the locus, whereas if $d = \pi/2 \pmod{\pi}$ we have merely the straight line

$$(\bar{y}x) = 0.$$

We shall call the general locus a "hypercircle."

THEOREM 8. *The totality of points at a given distance from a given point will depend upon three real parameters, except in the case where the distance is zero and there is but one point in the totality, or when, in the elliptic metric it is equal to* $\pi/2 \pmod{\pi}$ *and the totality consists in the points of a line. In the general case the equation of the locus is expressed by equating a Hermitian form to zero.*

If we place the point (1, 0, 0) at the center of the hypercircle, the equation takes one of the canonical forms

$$- \tan^2 d x_0\bar{x}_0 + x_1\bar{x}_1 + x_2\bar{x}_2 = 0, \qquad (11e)$$

$$x\bar{x} + y\bar{y} = r^2. \qquad (11p)$$

The polar of a given point with regard to the hypercircle will have the equation

$$- \tan^2 d\bar{y}_0 x_0 + \bar{y}_1 x_1 + \bar{y}_2 x_2 = 0, \qquad (12e)$$

$$\bar{x}'x + \bar{y}'y = r^2. \qquad (12p)$$

THEOREM 9. *The polar of a point with regard to a hypercircle is perpendicular to the line connecting the given point with the center. In the elliptic case the product of the tangents of the distances of the center of a hypercircle from a point and from its polar is equal to the square of the tangent of the radius, in the parabolic case the product of the distances is the radius squared.*

THEOREM 10. *If the pole of a line be at more than a radius distance from the center, the line meets the locus in a chain of points at the same distance from the foot of the perpendicular; if a point be on a hypercircle, its polar meets the hypercircle in that point and nowhere else, if the point be at less than a radius distance from the center, its polar contains no point of the hypercircle.*

A line meeting a hypercircle in a single point may be defined as a "tangent" thereto. It is not, however, the limiting position towards which a secant necessarily approaches as two points of intersection tend to coalesce. If we remember that the formulas for distance and angle are entirely analogous in the elliptic case, and that the tangential equation of a hypercircle is obtained from its point equation by exactly the same process as is used for a circle, we reach:

THEOREM 11*e*. *In the elliptic Hermitian metric, the lines which meet a*

fixed line at a fixed angle, which is not a right angle, will be tangent to a hypercircle whose center is the pole of the fixed line.

THEOREM 11p. *In the parabolic Hermitian metric the lines which meet a fixed line at a fixed angle other than a right angle will be parallel to the lines of a chain.*

Suppose that we have an equation obtained by setting a Hermitian form of non-vanishing discriminant equal to zero

$$\sum_{i,j=0}^{i,j=2} a_{ij}x_i\bar{x}_j = 0, \qquad |a_{ij}| \neq 0, \qquad a_{ji} = \bar{a}_{ij}. \tag{13}$$

If there be a single point whose coördinates satisfy this equation, there will be a system depending on three real parameters. We shall call the corresponding locus a "hyperconic." We define as the polar of a point (y) the line

$$\Sigma a_{ij}\cdot x_i\bar{y}_j = 0.$$

If (y) be a point of the hyperconic, it lies on its polar, which will contain no other point of the hyperconic, and which we shall call a "tangent" thereto. The tangential equation of the hyperconic is

$$\Sigma A_{ij}\cdot u_i\bar{u}_j = 0.$$

Let us seek a canonical form for the equation. We write the characteristic equation for the elliptic case

$$\begin{vmatrix} a_{00}-\rho & a_{01} & a_{02} \\ a_{01} & a_{11}-\rho & a_{12} \\ \bar{a}_{02} & \bar{a}_{12} & a_{22}-\rho \end{vmatrix} = 0.$$

This equation has surely one root, so that there is one point which has the same polar with regard to the given Hermitian form and with regard to the form which is the basis of our elliptic measurement. Taking this as (1, 0, 0) and giving to its common polar with regard to these two forms these same coördinates, the equation of the hyperconic becomes

$$a_{00}x_0\bar{x}_0 + a_{11}x_1\bar{x}_1 + a_{12}x_1\bar{x}_2 + \bar{a}_{12}\bar{x}_1x_2 + a_{22}x_2\bar{x}_2 = 0.$$

Consider the reduced characteristic equation

$$\begin{vmatrix} a_{11}-\rho & a_{12} \\ \bar{a}_{12} & a_{22}-\rho \end{vmatrix} = 0.$$

This will have equal roots if

$$(a_{11} - a_{22})^2 + 4\,a_{12}\bar{a}_{12} = 0,$$

two equations which can be satisfied only if

$$a_{11} = a_{22}, \qquad a_{12} = \bar{a}_{12} = 0,$$

and we have the hypercircle

$$a_{00}x_0\bar{x}_0 + a_{11}(x_1\bar{x}_1 + x_1\bar{x}_2) = 0.$$

In the general case this equation will have at least one root different from a_{00}, giving a second point with the same polar with regard to the two Hermitian forms. Taking this pole and its polar as (0, 1, 0) we find the canonical form for our hyperconic

$$\Sigma A_i x_i \bar{x}_i = 0. \tag{14}$$

Since this equation must not be an absurdity, we may assume that two of the A_i's are positive, and the third negative. Each vertex of the coördinate triangle is a "center" in the sense that a line through it which meets the hyperconic does so in pairs of points equidistant from the center, and on a normal chain therewith:

THEOREM 11*e*. *In the elliptic Hermitian metric each hyperconic has one in-center whose polar does not meet the locus, and two out-centers whose polars meet it in chains. The hyperconic is its own reflection in each of the centers.*

There is a little more variety in the possible forms of hyperconic in the parabolic case. We have two possibilities

(*A*)
$$\begin{vmatrix} a_{11} & a_{12} \\ \bar{a}_{12} & a_{22} \end{vmatrix} \neq 0.$$

Our hyperconic has the line equation

$$\Sigma A_{ij} u_i \bar{u}_j = 0:$$

we consider it at the same time as the form

$$u_1\bar{u}_1 + u_2\bar{u}_2 = 0.$$

The characteristic equation is

$$\begin{vmatrix} A_{00} - \rho & A_{01} & A_{02} \\ \bar{A}_{01} & A_{11} - \rho & A_{12} \\ \bar{A}_{02} & \bar{A}_{12} & A_{22} - \rho \end{vmatrix} = 0, \quad A_{00} \neq 0.$$

There will be at least one line other than (1, 0, 0) which has the same pole with regard to the two Hermitian forms. We call this and its pole (0, 1, 0) while the pole of the infinite line (1, 0, 0) with regard to the hyperconic shall be called the point (1, 0, 0). We are thus enabled to make exactly the same reductions as before, and reach the canonical form (14)

(*b*)
$$\begin{vmatrix} a_{11} & a_{12} \\ \bar{a}_{12} & a_{22} \end{vmatrix} = A_{00} = 0.$$

The point $(0,\ a_{22},\ -\ a_{12})$ has for its polar $x_0 = 0$ the infinite line. By a rotation of the plane we may force $(0,\ 1,\ 0)$ into this disagreeable rôle. There will be another point of the locus of such a nature that its tangent is perpendicular to the line connecting it with $(0,\ 1,\ 0)$ and this we take for $(1,\ 0,\ 0)$. We thus get the canonical forms

$$
\begin{aligned}
a_{01}x_0\bar{x}_1 + \bar{a}_{01}\bar{x}_0x_1 + A_2x_2\bar{x}_2 &= 0,\\
y\bar{y} &= \alpha x + \alpha\bar{x}.
\end{aligned}
\tag{15h}
$$

Note that all points on this locus are equidistant from the "focus" $(\alpha/2,\ 0)$ and the directrix $x = -\ \alpha/2$.

THEOREM 11p. *In the parabolic Hermitian metric a hyperconic has either one center, or else it meets the infinite line in a single point. In this latter case it is the locus of points equidistant from a given point and a given line. This line, the directrix, is the locus of the reflection of the focus in a tangent. Mutually perpendicular lines through the focus are conjugate with regard to the hyperconic, and tangents from a point on the directrix are mutually perpendicular in pairs.*

The usual geometric proofs for the parabola are applicable here. Suppose that a hyperconic has an out-center at the point $x_h = 1$ $x_k = x_l = 0$. Its equation may then be written

$$\beta_h x_h\bar{x}_h + \beta_k x_k\bar{x}_k - \beta_l x_l\bar{x}_l = 0.$$

The tangents from the out-center, which we call "asymptotes" will generate the variety

$$\beta_k x_k\bar{x}_k - \beta_l x_l\bar{x}_l = 0.$$

We may write our hyperconic parametrically in the form:

$$x_h = \frac{e^{i\Phi}}{\sqrt{\beta_h}}, \qquad x_k = \frac{\sinh A e^{i\Psi}}{\sqrt{\beta_k}}, \qquad x_l = \frac{\cosh A e^{iX}}{\sqrt{\beta_l}}.$$

The distance from this point to the asymptote

$$\frac{e^{i\Psi}}{\sqrt{\beta_e}}x_k - \frac{e^{iX}}{\sqrt{\beta_{12}}}x_l = 0$$

is

$$\frac{\sinh A - \cosh A}{\sqrt{P + Q\cosh^2 A + R\sinh^2 A}}$$

an expression which approaches zero as A becomes infinite. Exactly similar reasoning holds in the parabolic case

THEOREM 12. *If a point of a central hyperconic recede indefinitely from an out-center, its distance from the nearest asymptote through that center becomes infinitely small.*

We mentioned earlier that in the case of the non-central parabolic hyperconic, there was one point, called the focus, through which conjugate lines were mutually perpendicular. Let us see if there are corresponding points in the case of the central hyperconics. If there be any such point, the line connecting it with a center must be an axis, or else perpendicular to an axis through the given point. In any case, the point lies on an axis. Writing the hyperconic

$$\Sigma A_i x_i \bar{x}_i = 0, \tag{14}$$

we call our assumed focus $(0, y_k, y_l)$. An arbitrary line through this point will have the coördinates $(u_h, y_l, -y_k)$. The perpendicular thereto will be $(-(y_k\bar{y}_k + y_e\bar{y}_e), \bar{u}_h y_l, -\bar{u}_h y_k)$. The two will be conjugate if

$$\frac{-(y_k\bar{y}_k + y_l\bar{y}_l)}{A_h} + \frac{y_l\bar{y}_l}{A_k} + \frac{y_k\bar{y}_k}{A_l} = 0,$$

$$y_h = 0, \qquad y_k = \sqrt{A_l(A_h - A_k)}e^{i\Theta}, \qquad y_l = \sqrt{A_k(A_l - A_h)}e^{i\Phi}.$$

Of the three coefficients A_i one must be negative; we assume the other two positive, writing

$$\frac{y_k\bar{y}_k}{A_l(A_h - A_k)} = \frac{y_l\bar{y}_l}{A_k(A_l - A_h)} = 1.$$

We see that the negative coefficient can not be A_h, in fact A_h must be the numerically larger of the two positive coefficients. Every point so obtained shall be called a "focus," its polar being the directrix. We then find by a direct calculation,

THEOREM 13. *On one of the axes which does not connect two out-centers of a central hyperconic, there is a chain of foci. Conjugate lines through a focus are mutually perpendicular. In the elliptic Hermitian metric, the ratio of the sines of the distances of a point of a hyperconic from a focus and from the corresponding directrix is the same, not only for all foci, but for all points of the hyperconic. In the parabolic case it is the distances themselves which have a constant ratio.*

It is occasionally advantageous to write the equation of a hyperconic in a general symbolic form in the Clebsch-Aronhold notation. For instance, if the line equation be

$$u_a \bar{u}_{\bar{a}} = 0,$$

while the condition for perpendicularity is

$$u_\beta \bar{u}_{\bar{\beta}}' = \bar{u}_{\bar{\beta}} u_\beta' = 0,$$

and if (v) and (w) be two mutually perpendicular tangents to the hyperconic

$$v_\alpha \bar{v}_{\bar{\alpha}} = 0, \qquad w_\alpha \bar{w}_{\bar{\alpha}} = 0,$$

$$v_\beta \bar{w}_{\bar{\beta}} = \bar{v}_{\bar{\beta}} w_\beta = 0,$$

$$\begin{vmatrix} v_\alpha & v_\beta \\ w_\alpha & w_\beta \end{vmatrix} \cdot \begin{vmatrix} \bar{v}_{\bar{\alpha}} & \bar{v}_{\bar{\beta}} \\ \bar{w}_{\bar{\alpha}} & \bar{w}_{\bar{\beta}} \end{vmatrix} = 0,$$

$$|\alpha\beta x| \cdot |\bar{\alpha}\bar{\beta}\bar{x}| = 0.$$

THEOREM 14. *The locus of points whence tangents to a hyperconic are mutually perpendicular in pairs, is in the elliptic metric, another hyperconic with the same centers. In the parabolic case it is a hypercircle, or the directrix of a non-central hyperconic.*

We saw in theorem 7 that the usual trigonometric formulas hold for a triangle when, and only when, the altitudes are concurrent. It is not difficult to see, however, that the law of sines holds for every triangle. If then in the elliptic case, the vertices of a triangle be $A_1A_2A_3$ and the foot of the perpendicular from A_i upon the opposite side be A_i', we have

$$\sin (A_iA_i') = \sin (A_iA_j) \sin A_k = \sin (A_iA_k) \sin A_j,$$

$$\sin (A_iA_j) \sin (A_iA_k) \sin A_i = \sin (A_iA_i') \sin (A_jA_k) = \textit{const.}$$

We call this expression the "sine amplitude" of the triangle, and write it $\sin (A_1A_2A_3)$.* If the coördinates of the vertices be (y) (z) (t) we have

$$\sin (A_1A_2A_3) = \frac{\sqrt{|yzt|}\sqrt{|\bar{y}\bar{z}\bar{t}|}}{\sqrt{(y\bar{y})}\sqrt{(z\bar{z})}\sqrt{(t\bar{t})}}.$$

Notice also

$$\frac{\sin A_i}{\sin (A_jA_k)} = \frac{\sin (A_1A_2A_3)}{\sin (A_jA_k) \sin (A_kA_i) \sin (A_iA_j)}.$$

In the parabolic case we may take as double the measure of a triangle

$$(A_iA_j)(A_iA_k) \sin A_i = (A_iA_i')(A_jA_k).$$

In terms of non-homogeneous coördinates this is

$$\sqrt{\begin{vmatrix} x & y & 1 \\ x' & y' & 1 \\ x'' & y'' & 1 \end{vmatrix}} \sqrt{\begin{vmatrix} \bar{x} & \bar{y} & 1 \\ \bar{x}' & \bar{y}' & 1 \\ \bar{x}'' & \bar{y}'' & 1 \end{vmatrix}}.$$

We see, however, that a transversal through the vertex of a triangle will divide the triangle into two others whose measures add up to the measure of the given triangle only when the point of division belongs to the normal

* Conf. the Author's "Elements of Non-Euclidean Geometry," Oxford, 1909, p. 170.

chain of the two vertices collinear therewith. Hence we do not readily arrive at our measure by a double integration, and the whole subject seems to me of secondary interest.

4. **Differential formulas.** The fundamental differential expression which interests us is the squared distance element which is given in the elliptic case by the equation

$$ds^2 = \frac{(x\bar{x})(dxd\bar{x}) - (xd\bar{x})(\bar{x}dx)}{(x\bar{x})^2}. \tag{15e}$$

We have similarly the angular element

$$d\theta^2 = \frac{(u\bar{u})(dud\bar{u}) - (ud\bar{u})(\bar{u}du)}{(u\bar{u})^2}.$$

It is frequently better to drop the homogeneous point and line coordinates in differential work. Thus, assuming that a line does not pass through the point (1, 0, 0), we write its equation

$$ux + vy + 1 = 0,$$

and have for our differential forms

$$ds^2 = \frac{dxd\bar{x} + dyd\bar{y} + (xdy - ydx)(\bar{x}d\bar{y} - \bar{y}d\bar{x})}{(x\bar{x} + y\bar{y} + 1)^2}, \tag{16e}$$

$$d\theta^2 = \frac{dud\bar{u} + dvd\bar{v} + (udv - vdu)(\bar{u}d\bar{v} - \bar{v}d\bar{u})}{(u\bar{u} + v\bar{v} + 1)^2}. \tag{17e}$$

In the parabolic case we have the analogous but simpler expressions

$$dx^2 = dxd\bar{x} + dyd\bar{y}, \tag{16h}$$

$$d\theta^2 = \frac{(udv - vdu)(\bar{u}d\bar{v} - \bar{v}d\bar{u})}{(u\bar{u} + v\bar{v})^2}. \tag{17h}$$

Suppose that we have given a curve

$$y = y(x), \qquad \bar{y} = \bar{y}(\bar{x}).$$

The equation of the tangent is

$$\frac{y'}{y - xy'}X - \frac{1}{y - xy'}Y + 1 = 0,$$

$$du = \frac{yy''}{(y - xy')^2}dx, \qquad dv = \frac{-xy''}{(y - xy')^2}dx,$$

$$d\bar{u} = \frac{\bar{y}\bar{y}''}{(\bar{y} - \bar{x}\bar{y}^1)^2}d\bar{x}, \qquad d\bar{v} = \frac{-\bar{x}\bar{y}''}{(\bar{y} - \bar{x}\bar{y}^1)^2}d\bar{x},$$

$$ds^2 = \frac{1 + y'\bar{y}' + (xy' - xy')(\bar{y} - \bar{x}\bar{y}')}{(x\bar{x} + y\bar{y} + 1)^2}\, dx d\bar{x}:$$

$$d\theta^2 = \frac{(x\bar{x} + y\bar{y} + 1)y''\bar{y}''dxd\bar{x}}{[1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')]^2}.$$

We have, thus as an expression for the curvature

$$\frac{1}{k} = \frac{d\theta}{ds} = \frac{\sqrt{y''\bar{y}''}}{\left[\frac{1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')}{x\bar{x} + y\bar{y} + 1}\right]^{\frac{3}{2}}}. \tag{18e}$$

Suppose, secondly we consider a complex surface where the distance element is

$$ds^2 = 0dx^2 + 2Fdxd\bar{x} + 0d\bar{x}^2 = \frac{1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')}{(x\bar{x} + y\bar{y} + 1)^2}\, dxd\bar{x}.$$

We may write this a little more simply in the form

$$2F = \frac{u\bar{u} + v\bar{v} + 1}{v\bar{v}(x\bar{x} + y\bar{y} + 1)^2}.$$

We now look for the Gaussian curvature of this surface

$$\frac{\partial^2 \log F}{\partial^2 x d\bar{x}} = \left[\frac{\partial^2 \log (u\bar{u} + v\bar{v} + 1)}{\partial x \partial \bar{x}} - 2\frac{\partial^2 \log (x\bar{x} + y\bar{y} + 1)}{\partial x \partial \bar{x}}\right]$$

$$= \left[\frac{u'\bar{u}' + v'\bar{v}' + (uv' - vu')(\bar{u}\bar{v}' - \bar{v}\bar{u}')}{(u\bar{u} + v\bar{v} + 1)^2} - 2\frac{1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{y}\bar{x}')}{(x\bar{x} + y\bar{y} + 1)^2}\right]$$

$$= \left[\frac{(1 + x\bar{x} + y\bar{y})y''\bar{y}''}{[1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')]^2} - 2\frac{1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')}{(x\bar{x} + y\bar{y} + 1)^2}\right],$$

$$\frac{1}{K} = -\frac{1}{F}\frac{\partial^2 \log F}{\partial x \partial \bar{x}} = -\frac{2}{k^2} - 4. \tag{19e}$$

THEOREM 14. *The Gaussian curvature of a surface having the same distance element as a given curve is, in the elliptic case, 4 less than minus two times the square of the curvature of the given curve. In the parabolic case the difference of 4 between the two expressions is lacking.*

We next look for a curve of constant curvature

$$\frac{y''\bar{y}''}{\left[\frac{1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')}{x\bar{x} + y\bar{y} + 1}\right]^3} = C.$$

Assuming C not to be zero, we treat x and $\bar{x}$ as independent variables and differentiate the logarithms of both sides successively to x and to $\bar{x}$

$$\frac{y''\bar{y}''}{\left[\dfrac{1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')}{x\bar{x} + y\bar{y} + 1}\right]^3} = -2.$$

This is an absurd equation, hence we must have $C = 0$.

THEOREM 16. *The only curves of constant curvature in Hermitian metrics are straight lines.*

Let us look for a "geodesic thread" on a given curve, i.e., a system of points depending upon one real parameter which gives the shortest path between two of its members, the normal chain is a good example. We assume as before that y is a known function of x and $\bar{y}$ the conjugate imaginary function of $\bar{x}$. Let us find two functions $x = x(t)$ $\bar{x} = \bar{x}(t)$ which will minimize the integral

$$\int_a^b \sqrt{\frac{[1 + y'\bar{y}' + (y - xy')(\bar{y} - \bar{x}\bar{y}')]x'\bar{x}'dt}{(x\bar{x} + y\bar{y} + 1)^2}}.$$

We call this

$$\int_a^b R(x\bar{x}x'\bar{x}')dt = \int_a^b \sqrt{F(x, \bar{x}) \cdot x'\bar{x}'}dt.$$

To minimize this we may treat x and $\bar{x}$ as independent

$$\frac{\partial R}{\partial x} = \frac{d}{dt}\left(\frac{\partial R}{\partial x'}\right), \qquad \frac{\partial R}{\partial \bar{x}} = \frac{d}{dt}\left(\frac{\partial R}{\partial x'}\right).$$

These reduce to the single equation

$$x'\frac{\partial F}{\partial x} - \bar{x}'\frac{\partial F}{\partial \bar{x}} = \frac{F}{x'\bar{x}'}(x'\bar{x}'' - \bar{x}'x'').$$

Let

$$\phi = \log F, \qquad x'\frac{\partial \phi}{\partial x} - \bar{x}'\frac{\partial \phi}{\partial \bar{x}} = \frac{x'\bar{x}'' - \bar{x}'x''}{x'\bar{x}'}.$$

Let

$$x = U + iV, \qquad \bar{x} = U - iV,$$

$$x' = U' + iV', \qquad \bar{x}' = U' - iV',$$

$$V'\frac{\partial \phi}{\partial U} - U'\frac{\partial \phi}{\partial V} = \frac{2(V'U'' - U'V'')}{U'^2 + V'^2}$$

$$\frac{dV}{dU} = \frac{V'}{U'}, \qquad V' = U'\frac{dV}{dU},$$

$$U'^3\frac{d^2V}{dU^2} = U'V'' - V'U'',$$

$$\frac{d^2V}{dU^2} + \frac{1}{2}\left[1 + \left(\frac{dV}{dU}\right)^2\right]\left[\frac{dV}{dU}\frac{\partial \phi}{\partial U} - \frac{\partial \phi}{\partial V}\right] = 0.$$

In the case of the line $y = 0$,

$$F = \frac{1}{(x\bar{x} + 1)^2},$$

$$(x\bar{x} + 1)(x'\bar{x}'' - \bar{x}'x'') + 2x'\bar{x}'(\bar{x}x' - x\bar{x}') = 0.$$

Let

$$x = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad \bar{x} = \frac{\bar{\alpha}t + \bar{\beta}}{\bar{\gamma}t + \bar{\delta}},$$

$$\alpha\bar{\beta} + \gamma\bar{\delta} = \bar{\alpha}\beta + \bar{\gamma}\delta,$$

a normal chain.

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